Quantum string dynamics in the conformal invariant $SL(2,R)$ WZWN background: Anti–de Sitter space with torsion

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We consider classical and quantum strings in the conformally invariant background corresponding to the $SL(2,R)$ WZWN model. This background is locally anti–de Sitter spacetime with non-vanishing torsion. Conformal invariance is expressed as the torsion being parallelizing. The precise effect of the conformal invariance on the dynamics of both circular and generic classical strings is extracted. In particular, the conformal invariance gives rise to a repulsive interaction of the string with the background which precisely cancels the dominant attractive term arising from gravity. We perform both semi-classical and canonical string quantization, in order to see the effect of the conformal invariance of the background on the string mass spectrum. Both approaches yield that the high-mass states are governed by $m \sim HN$ ($N \in N_0$, N ''large''), where m is the string mass and H is the Hubble constant. It follows that the level spacing grows proportionally to N: $d(m^2\alpha')/dN \sim N$, while the string entropy goes like $S \sim \sqrt{m}$. Moreover, it follows that there is no Hagedorn temperature, so that the partition function is well defined at any positive temperature. All results are compared with the analogue results in anti–de Sitter spacetime, which is a nonconformal invariant background. Conformal invariance *simplifies* the mathematics of the problem but the physics remains mainly *unchanged*. Differences between conformal and non-conformal backgrounds only appear in the intermediate region of the string spectrum, but these differences are minor. For low and high masses, the string mass spectra in conformal and non-conformal backgrounds are identical. Interestingly enough, conformal invariance fixes the value of the spacetime curvature to be $-69/(26\alpha')$. [S0556-2821(98)00314-2]

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I. INTRODUCTION

The systematic investigation of string dynamics in curved spacetimes started in Ref. [1] has revealed new insight and new physical phenomena with respect to string propagation in flat spacetime (and with respect to quantum fields in curved spacetime) [2]. These results are relevant both for fundamental quantum strings and for cosmic strings, which behave in a classical way.

Cosmic strings can be considered in arbitrary curved spacetime backgrounds, while fundamental quantum strings demand a conformally invariant background for quantum consistency (conformal invariance is a necessary although not sufficient condition for consistency). However, most curved spacetimes that were historically of physical interest in general relativity and cosmology are not conformally invariant. On the other hand, certain group-manifolds and coset-spaces provide a large family of new spacetimes that are conformally invariant, but they are generally not so interesting from a physical point of view.

The classical and quantum string dynamics and their associated effects in a wide class of string backgrounds (conformal and non-conformal invariant) have been widely investigated by the present authors [1,2].

In this paper, we consider classical and quantum strings in the conformally invariant background corresponding to the $SL(2,R)$ Wess-Zumino-Witten-Novikov (WZWN) model. This background is locally anti–de Sitter spacetime with non-vanishing parallelizing torsion. The cosmological importance of anti–de Sitter spacetime is somewhat less than that of (say) de Sitter spacetime, but it is in any case an example of a Robertson-Walker spacetime. Moreover, after a suitable point identification, the background corresponds to the $2+1$ black hole (BH) anti–de Sitter (AdS) spacetime [3], which is a toy-model for investigations of black hole phenomena in higher dimensions. Thus, our interest in the $SL(2,R)$-WZWN background is due to a compromise of conformal invariance, physical interest and simplicity.

Many mathematical aspects of the $SL(2,R)$-WZWN model have been discussed in the literature (see for instance Refs. [4–7]), but we find that the physical aspects have not really been extracted so far. The purpose of this paper is to investigate directly the effect of the conformal invariance on the string dynamics, both classically and quantum mechanically. The conformal invariance is expressed via a parallelizing torsion. Thus we consider the string equations of motion in a background consisting of the standard anti–de Sitter metric plus an anti-symmetric tensor representing the parallelizing torsion. By considering special as well as generic solutions to these equations, and by comparing with the analogue results in the absence of torsion, we extract the *precise* effect of the conformal invariance on the dynamics of classical strings. Similarly, after quantization, we extract the effect of the conformal invariance on the quantum phenomena, especially on phenomena related to the quantum mass spectrum.

In the cases of AdS and BH-AdS spacetimes, the torsion corresponding to conformal invariance provides a repulsive term, which in the string dynamics precisely cancels the dominant attractive term arising from gravity.

As a general effect, we find that conformal invariance simplifies the mathematics of the problem; however, the physics is more or less *unchanged*. In fact, in the two limits $n \ll (H^2\alpha')^{-1}$ and $n \gg (H^2\alpha')^{-1}$, of the string mass spectrum the results obtained here are in *exact* agreement with the results obtained without torsion [16,22]. For small n and large n, the spectrum is not affected by the conformal invariance, while there are some minor changes in the intermediate region.

The frequencies of string oscillators are shifted away from integers n:

$$\omega_n = |n \pm mH\alpha'|, \tag{1.1}$$

while in $2+1$ AdS spacetime without torsion, the frequencies turned out to be [22]

$$\omega_n = \sqrt{n^2 + m^2H^2\alpha'^2} \quad \text{(without torsion)}. \tag{1.2}$$

In both cases the frequencies are real and the strings experience completely regular oscillatory behavior. Moreover, for small n ($n \ll mH\alpha'$) and large n ($n \gg mH\alpha'$), the results agree, while there is a minor difference in the intermediate region; in fact, from Eqs. (1.1) and (1.2) [see also Eqs. (5.35), (5.36)] follow that the effect of the conformal invariance is to "complete the square." This effect shows itself too in the mass spectrum [Eqs. (6.9) and (6.18)].

Notice that states with the same eigenvalue of the number-operator do not necessarily have the same mass [22]. This is the case both for the low-mass states and the high-mass states. In the low-mass spectrum, the effect is just like a fine-structure effect, while in the high-mass spectrum, the states are completely mixed up. This is very different from Minkowski spacetime where states with the same eigenvalue of the number-operator always have the same mass.

Interestingly enough, conformal invariance fixes the value of the spacetime curvature to be

$$R = -\frac{69}{26\alpha'}.$$

The paper is organized as follows. In Sec. II, we review the WZWN construction for the group $SL(2,R)$. We consider the two parametrizations corresponding to global $2+1$ anti–de Sitter spacetime and $2+1$ black hole anti–de Sitter spacetime, respectively. In both cases, we read off the metric and torsion.

In Sec. III, we solve the classical string equations of motion and constraints in the above mentioned backgrounds, for the special configuration describing an oscillating circular string. We compare with the analogue results obtained in the absence of torsion, and then extract and discuss the *precise* effect of the conformal invariance (expressed via the parallelizing torsion).

In Sec. IV, we perform a semi-classical quantization of the oscillating circular strings, obtained in Sec. III. In this way we obtain the semi-classical mass spectrum. Again, we compare with the analogue results obtained in the absence of torsion, and extract and discuss the *precise* effect of the conformal invariance.

In Sec. V, we consider more generic string configurations by solving the classical string equations of motion and constraints in a perturbative scheme. We compute first and second order string-fluctuations around the string center of mass, and derive the classical mass formula. The frequencies of string fluctuations are compared with the analogue results obtained in the absence of torsion.

In Sec. VI, we perform a canonical quantization of the oscillator modes, and we derive the quantum mass formula. The mass formula is investigated in detail in different regimes, and we compare with the results obtained using semi-classical quantization in Sec. IV. In particular, we derive the asymptotic level spacing and the entropy of string states.

Finally, in Sec. VII, we give our concluding remarks, and we discuss possible continuations of our work.

II. CLASSICAL EQUATIONS OF MOTION

To fix our notations and conventions, we give in this section a short review of the WZWN construction for the group $SL(2,R)$. This will lead to the classical string equations of motion in the background of $(2+1)$-dimensional anti–de Sitter (AdS) spacetime with the presence of parallelizing torsion. A different parametrization of the group manifold leads to the classical string equations of motion in the background of $2+1$ dimensional black hole anti–de Sitter (BH-AdS) spacetime [3] with the presence of parallelizing torsion.

Our starting point is the sigma-model action including the WZWN term at level k [8]:

$$S_\sigma = -\frac{k}{4\pi}\int_M d\tau d\sigma\, \eta^{\alpha\beta}\,\mathrm{Tr}[g^{-1}\partial_\alpha g g^{-1}\partial_\beta g] - \frac{k}{6\pi}\int_B \mathrm{Tr}[g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg]. \tag{2.1}$$

Here M is the boundary of the manifold B, and g is a group-element of $SL(2,R)$:

$$g = \begin{pmatrix} a & u \\ -v & b \end{pmatrix}, \quad ab + uv = 1. \tag{2.2}$$

Then, the action Eq. (2.1) takes the form [9]

$$S_\sigma = -\frac{k}{2\pi}\int_M d\tau d\sigma[\dot{a}\dot{b} - a'b' + \dot{u}\dot{v} - u'v'] - \frac{k}{\pi}\int_M d\tau d\sigma\, \log(u)[\dot{a}b' - a'\dot{b}], \tag{2.3}$$

where dot and prime denote derivative with respect to τ and σ, respectively. We shall now consider a parametrization of the group manifold corresponding to global $2+1$ AdS spacetime. We first introduce new coordinates (X,Y,W,T):

$$a=H(W+X),\quad b=H(W-X),$$
$$u=H(T-Y),\quad v=H(T+Y), \tag{2.4}$$

where H is a constant (the Hubble constant). Then we get, from Eq. (2.2),

$$X^2+Y^2-W^2-T^2=-\frac{1}{H^2}, \tag{2.5}$$

which is the standard embedding equation for $2+1$ AdS spacetime.

Using the standard parametrization (see for instance [10])

$$X=r\cos\varphi,\quad W=\frac{1}{H}\sqrt{1+H^2r^2}\cos Ht,$$
$$Y=r\sin\varphi,\quad T=\frac{1}{H}\sqrt{1+H^2r^2}\sin Ht, \tag{2.6}$$

the action Eq. (2.3) becomes

$$S_\sigma=-\frac{kH^2}{2\pi}\int_M d\tau d\sigma\left[-(1+H^2r^2)(t'^2-\dot{t}^2)+\frac{r'^2-\dot{r}^2}{1+H^2r^2}+r^2(\varphi'^2-\dot{\varphi}^2)\right]-\frac{kH^3}{\pi}\int_M d\tau d\sigma\, r^2[\dot{t}\varphi'-t'\dot{\varphi}]. \tag{2.7}$$

Let us recall that the generic sigma-model action in the presence of metric $g_{\mu\nu}$ and anti-symmetric tensor $B_{\mu\nu}$ is

$$S_\sigma=\frac{1}{2\pi\alpha'}\int_M d\tau d\sigma[g_{\mu\nu}(\dot{X}^\mu\dot{X}^\nu-X'^\mu X'^\nu)+2B_{\mu\nu}(\dot{X}^\nu X'^\mu-\dot{X}^\mu X'^\nu)]. \tag{2.8}$$

In our case, $X^\mu=(t,r,\varphi)$ and we can then read off:

$$g_{tt}=-(1+H^2r^2),\quad g_{rr}=(1+H^2r^2)^{-1},\quad g_{\varphi\varphi}=r^2,$$
$$B_{t\varphi}=-B_{\varphi t}=\frac{1}{2}Hr^2, \tag{2.9}$$

while the level of the WZWN model is related to the string tension and H through

$$k=(H^2\alpha')^{-1}. \tag{2.10}$$

Thus, the background is $2+1$ AdS spacetime in static coordinates (which cover AdS spacetime completely), plus an anti-symmetric tensor $B_{\mu\nu}$ with a single non-zero component $B_{t\varphi}$.

Alternatively we can parametrize the group-element Eq. (2.2) in the following way [11]:

$$a=\frac{Hr}{\sqrt{M}}e^{\sqrt{M}\varphi},\quad u=\pm\sqrt{\frac{H^2r^2-M}{M}}\,e^{H\sqrt{M}t},$$
$$b=\frac{Hr}{\sqrt{M}}e^{-\sqrt{M}\varphi},\quad v=\pm\sqrt{\frac{H^2r^2-M}{M}}\,e^{-H\sqrt{M}t}, \tag{2.11}$$

where M is a constant. This parametrization holds for $H^2r^2-M>0$, but analogue expressions hold for $H^2r^2-M<0$.

With this parametrization, the action Eq. (2.3) becomes

$$S_\sigma=-\frac{kH^2}{2\pi}\int_M d\tau d\sigma\left[-(H^2r^2-M)(t'^2-\dot{t}^2)+\frac{r'^2-\dot{r}^2}{H^2r^2-M}+r^2(\varphi'^2-\dot{\varphi}^2)\right]-\frac{kH^3}{\pi}\int_M d\tau d\sigma\, r^2[\dot{t}\varphi'-t'\dot{\varphi}]. \tag{2.12}$$

This is of course equivalent to the AdS-action (2.7) in the case $M=-1$. However, for positive M, the background corresponding to Eq. (2.12) is

$$g_{tt}=-(H^2r^2-M),\quad g_{rr}=(H^2r^2-M)^{-1},\quad g_{\varphi\varphi}=r^2,$$
$$B_{t\varphi}=\frac{1}{2}Hr^2,\quad B_{\varphi t}=-\frac{1}{2}Hr^2, \tag{2.13}$$

which is the $2+1$ BH-AdS spacetime [3] plus an anti-symmetric tensor $B_{\mu\nu}$ with a single non-zero component $B_{t\varphi}$. And again the level of the WZWN model is $k=(H^2\alpha')^{-1}$ [12]. We also recall that M is the mass of the black hole while H is the Hubble constant.

We close this section with some general remarks concerning the action Eq. (2.8). The corresponding equations of motion are

$$\ddot{X}^\mu-X''^\mu+\Gamma^\mu_{\rho\sigma}(\dot{X}^\rho\dot{X}^\sigma-X'^\rho X'^\sigma)+H^\mu_{\rho\sigma}(\dot{X}^\rho X'^\sigma-\dot{X}^\sigma X'^\rho)=0, \tag{2.14}$$

where, as usual, $H_{\mu\rho\sigma}=B_{\mu\rho,\sigma}-B_{\mu\sigma,\rho}+B_{\rho\sigma,\mu}$.

The string equations of motion should be supplemented by the constraints:

$$g_{\mu\nu}\dot{X}^\mu X'^\nu=0,\quad g_{\mu\nu}(\dot{X}^\mu\dot{X}^\nu+X'^\mu X'^\nu)=0. \tag{2.15}$$

It is convenient to introduce world-sheet light-cone coordinates:

$$\sigma^\pm=\tau\pm\sigma. \tag{2.16}$$

Then, Eq. (2.14) takes the compact form:

$$X^\lambda_-\overline{\nabla}_\lambda X^\mu_+=0, \tag{2.17}$$

where $\overline{\nabla}_\lambda$ is the generalized covariant derivative defined in terms of the generalized Christoffel symbol:

$$\overline{\Gamma}^{\mu}_{\rho\sigma}=\Gamma^{\mu}_{\rho\sigma}+H^{\mu}_{\rho\sigma}. \tag{2.18}$$

Notice that $\overline{\Gamma}^{\mu}_{\rho\sigma}$ is obviously not symmetric in the two lower indices.

III. CIRCULAR STRINGS. CLASSICAL DYNAMICS

To investigate the effect of the conformal invariance on the string dynamics, we shall first consider the special string configurations representing oscillating circular strings.

We consider the background Eq. (2.9), corresponding to the $2+1$ AdS spacetime; the results in the background Eq. (2.13), corresponding to the $2+1$ BH-AdS spacetime, can then be obtained immediately.

The ansatz describing oscillating circular strings is

$$t=t(\tau),\quad r=r(\tau),\quad \varphi=\sigma. \tag{3.1}$$

Then, Eqs. (2.14) and (2.15) lead to

$$\ddot{t}+\frac{2H^2r\dot{r}\dot{t}}{1+H^2r^2}+\frac{2Hr\dot{r}}{1+H^2r^2}=0, \tag{3.2}$$

$$\ddot{r}+(1+H^2r^2)H^2r\dot{t}^2+(1+H^2r^2)r-\frac{H^2r\dot{r}^2}{1+H^2r^2}+2(1+H^2r^2)Hr\dot{t}=0, \tag{3.3}$$

supplemented by the constraint

$$-(1+H^2r^2)\dot{t}^2+\frac{\dot{r}^2}{1+H^2r^2}+r^2=0. \tag{3.4}$$

These three equations (3.2)–(3.4) are consistently integrated to

$$\dot{t}=\frac{E-Hr^2}{1+H^2r^2}, \tag{3.5}$$

$$\dot{r}^2=-(1+2EH)r^2+E^2, \tag{3.6}$$

where E is a non-negative integration constant. Equation (3.6) can be conveniently written as

$$\dot{r}^2+V(r)=0;\quad V(r)=(1+2EH)r^2-E^2, \tag{3.7}$$

that is, the potential $V(r)$ is *quadratic* in r. This is a great simplification as compared to the case of AdS without torsion. In that case [13], the potential was *quartic* in r and given by

$$V(r)=(1+H^2r^2)r^2-E^2\quad \text{(without torsion)} \tag{3.8}$$

that is, the solution involved elliptic functions [13]. In the present case with conformal invariance, the solution is instead obtained in terms of trigonometric functions (see later). Thus, an effect of the conformal invariance is that the mathematics simplifies considerably. It is also interesting to notice that the torsion, corresponding to conformal invariance, gives rise to repulsion at large distances, while gravity itself gives rise to attraction in AdS spacetime. This follows from a comparison of the two potentials Eqs. (3.7) and (3.8). It is seen from these expressions that the parallelizing torsion provides the term $-H^2r^4$ for large r, i.e., a repulsive term in the potential. In fact, this repulsive term precisely cancels the dominant attractive term in the potential Eq. (3.8) in the absence of torsion. The final outcome, in the presence of conformal invariance, is that the potential, Eq. (3.7), is still attractive, but it is only *quadratic* in r.

As for the dynamics of the circular strings in the presence of conformal invariance, we see from Eq. (3.7) that for a given value of E (and fixed H), the string oscillates between $r=0$ and $r=r_{\max}$:

$$r_{\max}=\sqrt{\frac{E^2}{1+2EH}};\quad E\geqslant 0. \tag{3.9}$$

Notice also that $\dot{t}$ is always positive during the oscillations.

In the case of circular strings in the background of $2+1$ BH-AdS spacetime, Eq. (2.13), one finds in a similar way:

$$\dot{t}=\frac{E-Hr^2}{H^2r^2-M}, \tag{3.10}$$

$$\dot{r}^2=(M-2EH)r^2+E^2. \tag{3.11}$$

Then, the potential is:

$$\dot{r}^2+V(r)=0;\quad V(r)=-(M-2EH)r^2-E^2, \tag{3.12}$$

which is again *quadratic* in r.

In the $2+1$ BH-AdS spacetime, there is an event horizon at $r_{\rm hor}=\sqrt{M}/H$, and we demand that $\dot{t}\geqslant 0$ everywhere outside the horizon. This leads to the constraint on the integration constant E:

$$E>\frac{M}{H}. \tag{3.13}$$

It follows that for a given value of E fulfilling Eq. (3.13), a circular string has a maximal radius $r=r_{\max}$:

$$r_{\max}=\sqrt{\frac{E^2}{2EH-M}}, \tag{3.14}$$

it then contracts, crosses the horizon and falls into the black hole. Qualitatively, this is the same behavior as in the absence of torsion [13]. But also in this case of BH-AdS spacetime, the conformal invariance simplifies the mathematics. More precisely, as in the case of AdS spacetime, the torsion corresponding to conformal invariance provides a repulsive term, which precisely cancels the dominant attractive term obtained from gravity.

IV. CIRCULAR STRINGS. SEMI-CLASSICAL QUANTIZATION

In this section we perform a semi-classical quantization of the circular string configurations in the $2+1$ AdS spacetime, obtained in the previous section. We use an approach developed in field theory by Dashen *et al.* [14] (see also [15]), based on the stationary phase approximation of the functional integral. In our context, this is supposed to be a good approximation in the ''semi-classical'' regime where $H^2\alpha' \ll 1$.

The method can be only used for time-periodic solutions of the classical equations of motion. Thus it can be used for the oscillating circular strings in the $2+1$ AdS spacetime. On the other hand, the circular strings in the $2+1$ BH-AdS spacetime are not truely time-periodic because of the causal properties of the background: once the strings have passed the horizon, they will not re-appear (although the solutions are formally time-periodic from the mathematical point of view).

The result of the stationary phase integration is expressed in terms of the function $W(m)$ [14]:

$$W(m) \equiv S_{\mathrm{cl}}(T(m)) + mT(m), \tag{4.1}$$

where S_{cl} is the action of the classical solution, m is the mass and the period $T(m)$ is implicitly given by:

$$\frac{dS_{\mathrm{cl}}}{dT} = -m. \tag{4.2}$$

Here it is important that T is the period in a *physical* time variable. In our case, it will be the period in the target-space time t. The bound state quantization condition then becomes [14]:

$$W(m) = 2\pi n; \quad n \in N. \tag{4.3}$$

This condition is generally expected to hold for n ''large.'' In our case, this will correspond to (say) $n \gg H^2\alpha'$.

We now use this method on the oscillating circular strings in $2+1$ AdS spacetime, as described by Eqs. (3.5), (3.6). These equations are solved by:

$$r(\tau) = \frac{E}{\sqrt{1+2EH}} \left|\sin[\sqrt{1+2EH}\,\tau]\right|, \tag{4.4}$$

$$Ht(\tau) = \arctan\left(\frac{1+EH}{\sqrt{1+2EH}} \tan[\sqrt{1+2EH}\,\tau] \right) - \tau, \tag{4.5}$$

where we took initial conditions such that:

$$t(0) = 0, \quad r(0) = 0. \tag{4.6}$$

The period of the solution, which is twice the period of r, is in world-sheet time τ given by:

$$T_\tau = \frac{2\pi}{\sqrt{1+2EH}}. \tag{4.7}$$

The classical action over one period is obtained from Eq. (2.8), using Eq. (2.9) and Eqs. (4.4) and (4.5):

$$S_{\mathrm{cl}} = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \int_0^{T_\tau} d\tau [g_{\mu\nu}(\dot{X}^\mu \dot{X}^\nu - X'^\mu X'^\nu) + 2B_{\mu\nu}(\dot{X}^\nu X'^\mu - \dot{X}^\mu X'^\nu)] = -\frac{2}{\alpha'} \int_0^{T_\tau} d\tau \frac{(1+EH)r^2}{1+H^2r^2}$$
$$= -\frac{4\pi(1+EH)}{H^2\alpha'} \left[\frac{1}{\sqrt{1+2EH}} - \frac{1}{1+EH} \right]. \tag{4.8}$$

As explained after Eq. (4.2), the period T must be the period in the physical time t. This period is obtained from Eq. (4.7) and Eq. (4.5):

$$T_t = \frac{2\pi}{H} \left(1 - \frac{1}{\sqrt{1+2EH}} \right). \tag{4.9}$$

Then:

$$S_{\mathrm{cl}}(T_t) = -\frac{1}{2\pi\alpha'} T_t^2 \left(1 - \frac{HT_t}{2\pi} \right)^{-1}. \tag{4.10}$$

From Eq. (4.2) we can then obtain the mass:

$$m = -\frac{dS_{\mathrm{cl}}}{dT} = \frac{1}{2\pi\alpha'} \left(2T_t - \frac{HT_t^2}{2\pi} \right) \left(1 - \frac{HT_t}{2\pi} \right)^{-2}, \tag{4.11}$$

which can be inverted to obtain the physical period T_t:

$$T_t = 2\pi \frac{\sqrt{1+Hm\alpha'} - 1}{H\sqrt{1+Hm\alpha'}}. \tag{4.12}$$

Finally, the quantization condition Eqs. (4.1)–(4.3) becomes:

$$W(m) \equiv S_{\mathrm{cl}}(T(m)) + mT(m) = 2\pi n, \tag{4.13}$$

i.e.:

$$\frac{2\pi}{H^2\alpha'} (\sqrt{1+mH\alpha'} - 1)^2 = 2\pi n. \tag{4.14}$$

This equation can be solved for m giving:

$$\alpha' m^2 = 4n \left(1 + \frac{\sqrt{H^2\alpha' n}}{2} \right)^2, \tag{4.15}$$

which gives the spectrum of quantum string states.

Notice that for ''small'' n ($n \ll (H^2\alpha')^{-1}$) it gives:

$$\alpha' m^2 = 4n, \tag{4.16}$$

which is the Minkowski result, while for ''large'' n [$n \gg (H^2\alpha')^{-1}$]:

$$\alpha' m^2 = H^2\alpha' n^2. \tag{4.17}$$

These results must be compared with the analogue results obtained for circular strings in AdS spacetime but *without* including torsion [16]. As a general effect, we see that in the presence of conformal invariance, the mathematics is much simpler. However, the physics is more or less *unchanged*. In fact, in the two limits $n \ll (H^2\alpha')^{-1}$ and $n \gg (H^2\alpha')^{-1}$, the results obtained here are in *exact* agreement with the results obtained without torsion [16]. That is, for small n and large n, the spectrum is not affected by the conformal invariance, while there may be some minor changes in the intermediate region.

V. PERTURBATIONS AROUND STRING CENTER OF MASS

The results in Secs. III and IV were obtained for special string configurations. In order to see if these results are more generic or just particular to the circular strings, we must consider more general string configurations.

The equations of motion and constraints Eqs. (2.14), (2.15) can in principle be solved exactly in the case of $SL(2,R)$, since it is a group-manifold [17,18], but the formulas (see for instance [19]) are formal and not explicit enough for further investigations of the string dynamics. Instead, we shall use here the method of expansion around the string center of mass [1], that is, we will compute first and second order string fluctuations around the point-particle geodesic representing the center of mass of the string. In the first subsection, we consider a generic 3-D spacetime with arbitrary torsion. This subsection is thus the generalization of subsection III A in Ref. [13] to the case of a spacetime with torsion. Then, in the following subsection, we specialize to the case of $2+1$ AdS spacetime with parallelizing torsion.

A. General formalism

To be more precise, consider first the equations of motion Eq. (2.14); the constraints will be dealt with afterwards. We then expand [1]

$$X^\mu(\tau,\sigma) = q^\mu(\tau) + \eta^\mu(\tau,\sigma) + \xi^\mu(\tau,\sigma) + \cdots \tag{5.1}$$

where $q^\mu(\tau)$ represents the string center of mass, while $\eta^\mu(\tau,\sigma)$ and $\xi^\mu(\tau,\sigma)$ are the first and second order string perturbations, respectively.

After insertion into Eq. (2.14), the equations of motion are to be solved order by order in the expansion. To zeroth order we get:

$$\dot{q}^\lambda \nabla_\lambda \dot{q}^\mu = 0, \tag{5.2}$$

which is just the standard general relativity geodesic equation; obviously the torsion does not couple to the string center of mass. To first order in the expansion, we get after a little algebra the following equation for $\eta^\mu(\tau,\sigma)$:

$$\dot{q}^\lambda \overline{\nabla}_\lambda (\dot{q}^\delta \overline{\nabla}_\delta \eta^\mu) - \overline{R}^\mu_{\sigma\rho\lambda}\dot{q}^\rho \dot{q}^\sigma \eta^\lambda - \eta''^\mu = 2H^\mu_{\rho\sigma}\dot{q}^\rho(\dot{q}^\delta \overline{\nabla}_\delta \eta^\sigma - \eta'^\sigma), \tag{5.3}$$

where $\overline{R}^\mu_{\sigma\rho\lambda}$ is the generalized curvature defined via the generalized Christoffel symbol Eq. (2.18):

$$\overline{R}^\lambda_{\mu\nu\beta} = \overline{\Gamma}^\lambda_{\mu\nu,\beta} - \overline{\Gamma}^\lambda_{\mu\beta,\nu} + \overline{\Gamma}^\alpha_{\mu\nu}\overline{\Gamma}^\lambda_{\alpha\beta} - \overline{\Gamma}^\alpha_{\mu\beta}\overline{\Gamma}^\lambda_{\alpha\nu}. \tag{5.4}$$

Notice that Eq. (5.3) is a special case of the generalized Raychaudhury equation for strings in the presence of torsion [20]. Moreover, if we skip the η' and η'' terms in Eq. (5.3), then it is just the generalized geodesic deviation equation (see for instance [21]).

However, we can simplify Eq. (5.3) further. For a massive string, corresponding to the string center of mass satisfying

$$g_{\mu\nu}\dot{q}^\mu \dot{q}^\nu = -m^2\alpha'^2, \tag{5.5}$$

there are two physical polarizations of string perturbations around the geodesic $q^\mu(\tau)$ (we are in a 3-D spacetime). We therefore introduce two normal vectors n^μ_R $(R=1,2)$:

$$g_{\mu\nu}n^\mu_R \dot{q}^\nu = 0, \quad g_{\mu\nu}n^\mu_R n^\nu_S = \delta_{RS}, \tag{5.6}$$

and consider only first order perturbations in the form:

$$\eta^\mu = n^\mu_R \Phi^R, \tag{5.7}$$

where Φ^R are the comoving perturbations, i.e., the perturbations as seen by an observer travelling with the center of mass of the string. It must be noticed that for a string in a three dimensional spacetime, there is only one physical polarization of string perturbations (one transverse direction), but since our zeroth order solution is not a string but a point-particle, we get in some sense one polarization of perturbations too many at this stage. This extra degree of freedom will eventually have to be eliminated somehow using the constraints. Notice also that the normal vectors Eq. (5.6) are not uniquely defined. In fact, there is a gauge invariance originating from the freedom to make local rotations of the 2-bein spanned by the normal vectors. By generalizing the procedure of Ref. [13] to the case with torsion, we fix this gauge by taking the normal vectors to satisfy

$$\dot{q}^\mu \overline{\nabla}_\mu n^\nu_R = 0. \tag{5.8}$$

Using Eqs. (5.5)–(5.8) in Eq. (5.3), we find after contraction with $g_{\mu\nu}n^\nu_S$:

$$\ddot{\Phi}_S - \Phi''_S - \overline{R}_{\mu\sigma\rho\lambda}n^\mu_S n^\lambda_R \dot{q}^\rho \dot{q}^\sigma \Phi^R = 2H_{\mu\rho\sigma}\dot{q}^\rho n^\mu_S n^\sigma_R(\dot{\Phi}^R - \Phi'^R), \tag{5.9}$$

which for a given background $(g_{\mu\nu}, B_{\mu\nu})$ must be solved for Φ^R.

For the second order perturbations, the picture is a little more complicated since they couple also to the first order perturbations. We therefore consider the full set of perturbations ξ^μ:

$$\dot{q}^\lambda \overline{\nabla}_\lambda (\dot{q}^\delta \overline{\nabla}_\delta \xi^\mu) - \overline{R}^\mu_{\sigma\rho\lambda}\dot{q}^\rho \dot{q}^\sigma \xi^\lambda - \xi''^\mu - 2H^\mu_{\rho\sigma}\dot{q}^\rho(\dot{q}^\delta \overline{\nabla}_\delta \xi^\sigma - \xi'^\sigma) = U^\mu. \tag{5.10}$$

The term U^μ, which is bilinear in the first order perturbations, plays the role of a source and is explicitly given by:

$$U^\mu = -\Gamma^\mu_{\rho\sigma}(\dot{\eta}^\rho\dot{\eta}^\sigma - \eta'^\rho\eta'^\sigma) - 2H^\mu_{\rho\sigma}\dot{\eta}^\rho\eta'^\sigma - 2\Gamma^\mu_{\rho\sigma,\lambda}\dot{q}^\rho\eta^\lambda\dot{\eta}^\sigma - 2H^\mu_{\rho\sigma,\lambda}\dot{q}^\rho\eta^\lambda\eta'^\sigma - \frac{1}{2}\Gamma^\mu_{\rho\sigma,\lambda\delta}\dot{q}^\rho\dot{q}^\sigma\eta^\lambda\eta^\delta. \tag{5.11}$$

After solving Eqs. (5.9), (5.10) for the first and second order perturbations, the constraints Eq. (2.15) have to be imposed. In world-sheet light cone coordinates $\sigma^\pm = \tau \pm \sigma$, the constraints take the form:

$$T_{\pm\pm} = g_{\mu\nu}\partial_\pm X^\mu \partial_\pm X^\nu = 0. \tag{5.12}$$

The world-sheet energy-momentum tensor $T_{\pm\pm}$ is conserved, as can be easily verified using Eq. (2.14), and therefore can be written:

$$T_{--} = \frac{1}{2\pi}\sum_n \tilde{L}_n e^{-in(\sigma-\tau)},$$

$$T_{++} = \frac{1}{2\pi}\sum_n L_n e^{-in(\sigma+\tau)}. \tag{5.13}$$

At the classical level, the constraints are then simply:

$$L_n = \tilde{L}_n = 0; \quad n \in Z. \tag{5.14}$$

The quantum constraints will be considered in Sec. VI. Up to second order in the expansion around the string center of mass we find:

$$T_{\pm\pm} = -\frac{1}{4}m^2\alpha'^2 + g_{\mu\nu}\dot{q}^\mu\partial_\pm\eta^\nu + \frac{1}{4}g_{\mu\nu,\rho}\dot{q}^\mu\dot{q}^\nu\eta^\rho + g_{\mu\nu}\dot{q}^\mu\partial_\pm\xi^\nu + g_{\mu\nu}\partial_\pm\eta^\mu\partial_\pm\eta^\nu + g_{\mu\nu,\rho}\dot{q}^\mu\eta^\rho\partial_\pm\eta^\nu + \frac{1}{4}g_{\mu\nu,\rho}\dot{q}^\mu\dot{q}^\nu\xi^\rho + \frac{1}{8}g_{\mu\nu,\rho\sigma}\dot{q}^\mu\dot{q}^\nu\eta^\rho\eta^\sigma. \tag{5.15}$$

Formally, this is the same expression as in the absence of torsion, but one should keep in mind that the solutions for η and ξ involve the torsion and are different now.

Notice also that all results derived in this subsection hold for arbitrary torsion (not necessarily parallelizing). In the next subsection we apply the above formalism to the case of strings in the $2+1$ AdS spacetime with parallelizing torsion.

B. Strings in 3-D AdS spacetime with parallelizing torsion

We now consider strings in the $2+1$ AdS spacetime with conformal invariance, as described by the metric and torsion Eq. (2.9). For simplicity we consider a string with radially moving center of mass:

$$t = t(\tau), \quad r = r(\tau), \quad \varphi = \text{const.} \tag{5.16}$$

Then, Eqs. (5.2), (5.5) are integrated to:

$$\dot{t} = \frac{E}{1+H^2r^2}, \tag{5.17}$$

$$\dot{r}^2 = E^2 - (1+H^2r^2)m^2\alpha'^2, \tag{5.18}$$

where E is an integration constant [not the same as in Eqs. (3.5), (3.6)]. These equations are solved in terms of trigonometric functions, but we shall not need the explicit expressions here. A pair of independent normal-vectors satisfying Eq. (5.6) is provided by:

$$N^\mu_\perp = \left(0,0,\frac{1}{r}\right),$$

$$N^\mu_\parallel = \left(\frac{\dot{r}}{m\alpha'(1+H^2r^2)}, \frac{E}{m\alpha'}, 0\right). \tag{5.19}$$

However, they do not satisfy the gauge-condition Eq. (5.8). We therefore make a local rotation and define normal-vectors (n_1^μ, n_2^μ), satisfying also Eq. (5.8), by:

$$\begin{pmatrix} n_1^\mu \\ n_2^\mu \end{pmatrix} = \begin{pmatrix} \cos(mH\alpha'\tau) & -\sin(mH\alpha'\tau) \\ \sin(mH\alpha'\tau) & \cos(mH\alpha'\tau) \end{pmatrix} \begin{pmatrix} N^\mu_\perp \\ N^\mu_\parallel \end{pmatrix}. \tag{5.20}$$

Moreover, for the background Eq. (2.9):

$$\overline{R}^\lambda_{\mu\nu\beta} = 0, \tag{5.21}$$

which expresses the fact that the torsion is parallelizing for a group manifold (see for instance [21]).

Then, Eq. (5.9) for the first order perturbations reduces to:

$$\ddot{\Phi}_1 - \Phi_1'' + 2mH\alpha'(\dot{\Phi}_2 - \Phi_2') = 0, \tag{5.22}$$

$$\ddot{\Phi}_2 - \Phi_2'' - 2mH\alpha'(\dot{\Phi}_1 - \Phi_1') = 0. \tag{5.23}$$

Considering closed strings, we Fourier expand:

$$\Phi_R = \sum_n \phi_{Rn} e^{-in\sigma}; \quad R = 1,2, \tag{5.24}$$

so that Eqs. (5.22), (5.23) become:

$$\begin{pmatrix} \ddot{\phi}_{1n} \\ \ddot{\phi}_{2n} \end{pmatrix} + 2\mathcal{A}\begin{pmatrix} \dot{\phi}_{1n} \\ \dot{\phi}_{2n} \end{pmatrix} + \mathcal{B}\begin{pmatrix} \phi_{1n} \\ \phi_{2n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{5.25}$$

that is, two coupled ordinary linear differential equations of second order with constant (matrix) coefficients $\mathcal{A}$,$\mathcal{B}$:

$$\mathcal{A} = mH\alpha'\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} n^2 & 2inmH\alpha' \\ -2inmH\alpha' & n^2 \end{pmatrix}. \tag{5.26}$$

The first order τ-derivatives in Eq. (5.25) are eliminated by a rotation similar to Eq. (5.20):

$$\begin{pmatrix} \phi_{1n} \\ \phi_{2n} \end{pmatrix} = \begin{pmatrix} \cos(mH\alpha'\tau) & -\sin(mH\alpha'\tau) \\ \sin(mH\alpha'\tau) & \cos(mH\alpha'\tau) \end{pmatrix} \begin{pmatrix} \hat{\phi}_{1n} \\ \hat{\phi}_{2n} \end{pmatrix}, \tag{5.27}$$

such that:

$$\begin{pmatrix} \ddot{\hat{\phi}}_{1n} \\ \ddot{\hat{\phi}}_{2n} \end{pmatrix} + \begin{pmatrix} n^2+m^2H^2\alpha'^2 & 2inmH\alpha' \\ -2inmH\alpha' & n^2+m^2H^2\alpha'^2 \end{pmatrix} \begin{pmatrix} \hat{\phi}_{1n} \\ \hat{\phi}_{2n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{5.28}$$

and the equations are then decoupled by a unitary transformation:

$$\begin{pmatrix} \hat{\phi}_{1n} \\ \hat{\phi}_{2n} \end{pmatrix} = U \begin{pmatrix} C_{1n} \\ C_{2n} \end{pmatrix}; \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}. \tag{5.29}$$

Then, we get:

$$\begin{pmatrix} \ddot{C}_{1n} \\ \ddot{C}_{2n} \end{pmatrix} + \begin{pmatrix} (n+mH\alpha')^2 & 0 \\ 0 & (n-mH\alpha')^2 \end{pmatrix} \begin{pmatrix} C_{1n} \\ C_{2n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{5.30}$$

which are solved by:

$$C_{1n} = A_{1n} e^{-i|n+mH\alpha'|\tau} + \tilde{A}_{1n} e^{i|n+mH\alpha'|\tau},$$
$$C_{2n} = A_{2n} e^{-i|n-mH\alpha'|\tau} + \tilde{A}_{2n} e^{i|n-mH\alpha'|\tau}, \tag{5.31}$$

where $(A_{Rn}, \tilde{A}_{Rn})$ are integration constants.

The final result for the first order comoving perturbations is then:

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \sum_n e^{-in\sigma} \begin{pmatrix} ie^{imH\alpha'\tau} & -ie^{-imH\alpha'\tau} \\ e^{imH\alpha'\tau} & e^{-imH\alpha'\tau} \end{pmatrix} \begin{pmatrix} C_{1n} \\ C_{2n} \end{pmatrix}. \tag{5.32}$$

The constants $(A_{Rn}, \tilde{A}_{Rn})$ are constrained by the condition that (Φ_1, Φ_2) are real. This leads to:

$$\tilde{A}_{2n} = (A_{1-n})^\dagger, \quad A_{2n} = (\tilde{A}_{1-n})^\dagger. \tag{5.33}$$

As for the first order perturbations η^μ, we get:

$$\eta^\mu = \frac{1}{\sqrt{2}} \sum_n e^{-in\sigma} [(N_\parallel^\mu + iN_\perp^\mu) C_{1n} + (N_\parallel^\mu - iN_\perp^\mu) C_{2n}], \tag{5.34}$$

in terms of the normal-vectors Eq. (5.19) and the oscillators Eq. (5.31). This concludes the derivation of the first order perturbations. Notice that the frequencies are shifted away from integers n:

$$\omega_n = |n \pm mH\alpha'|, \tag{5.35}$$

and that the frequencies are different in the two directions perpendicular to the geodesic of the string center of mass. It is interesting to compare with the similar result in $2+1$ AdS spacetime but without torsion [22]. In that case, the frequencies of the first order perturbations turned out to be [22]:

$$\omega_n = \sqrt{n^2 + m^2H^2\alpha'^2} \quad (\text{without torsion}). \tag{5.36}$$

Thus, in both cases the frequencies are real, and therefore the strings experience completely regular oscillatory behavior. Moreover, for small n $(n \ll mH\alpha')$ and large n $(n \gg mH\alpha')$, the results agree, while there is a minor difference in the intermediate region; in fact, from Eqs. (5.35), (5.36) follow that the effect of the conformal invariance is to ''complete the square.''

We now come to the second order perturbations ξ^μ, as determined by Eqs. (5.10), (5.11). The computations are now going to be somewhat more complicated so we merely give the results of the different steps. We first re-define the ξ's and the corresponding sources U:

$$\xi^t = \hat{\xi}^t, \quad \xi^r = (1+H^2r^2)\hat{\xi}^r, \quad \xi^\phi = \frac{1}{r}\hat{\xi}^\varphi, \tag{5.37}$$

$$U^t = \hat{U}^t, \quad U^r = (1+H^2r^2)\hat{U}^r, \quad U^\phi = \frac{1}{r}\hat{U}^\varphi. \tag{5.38}$$

Equation (5.10) then takes the form:

$$\begin{pmatrix} \ddot{\hat{\xi}}^t \\ \ddot{\hat{\xi}}^r \\ \ddot{\hat{\xi}}^\varphi \end{pmatrix} - \begin{pmatrix} \hat{\xi}''^t \\ \hat{\xi}''^r \\ \hat{\xi}''^\varphi \end{pmatrix} + 2\mathcal{D} \begin{pmatrix} \dot{\hat{\xi}}^t \\ \dot{\hat{\xi}}^r \\ \dot{\hat{\xi}}^\varphi \end{pmatrix} + 2\mathcal{E} \begin{pmatrix} \hat{\xi}'^t \\ \hat{\xi}'^r \\ \hat{\xi}'^\varphi \end{pmatrix} + \mathcal{F} \begin{pmatrix} \hat{\xi}^t \\ \hat{\xi}^r \\ \hat{\xi}^\varphi \end{pmatrix} = \begin{pmatrix} \hat{U}^t \\ \hat{U}^r \\ \hat{U}^\varphi \end{pmatrix}, \tag{5.39}$$

where the matrices $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ are given by:

$$\mathcal{D} = \frac{H^2 r}{1+H^2r^2} \begin{pmatrix} \dot{r} & E & 0 \\ E & \dot{r} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.40}$$

$$\mathcal{E} = \frac{H}{1+H^2r^2} \begin{pmatrix} 0 & 0 & \dot{r} \\ 0 & 0 & E \\ (1+H^2r^2)\dot{r} & -(1+H^2r^2)E & 0 \end{pmatrix}, \tag{5.41}$$

$$\mathcal{F}=\frac{H^2}{1+H^2r^2}\begin{pmatrix} 0 & 2E\dot{r} & 0 \\ 0 & 2E^2-m^2\alpha'^2+H^2m^2r^2 & 0 \\ 0 & 0 & (1+H^2r^2)m^2\alpha'^2 \end{pmatrix}. \tag{5.42}$$

The first order τ-derivatives in Eq. (5.39) are eliminated by the transformation:

$$\begin{pmatrix} \hat{\xi}^t \\ \hat{\xi}^r \\ \hat{\xi}^\varphi \end{pmatrix}=\mathcal{G}\begin{pmatrix} \hat{\Sigma}^t \\ \hat{\Sigma}^r \\ \hat{\Sigma}^\varphi \end{pmatrix}; \quad \mathcal{G}=\mathrm{Exp}\left(-\int^\tau \mathcal{D}(\tau')d\tau'\right), \tag{5.43}$$

that is:

$$\mathcal{G}=\frac{-1}{(1+H^2r^2)m\alpha'}\begin{pmatrix} \dot{r} & E & 0 \\ E & \dot{r} & 0 \\ 0 & 0 & -(1+H^2r^2)m\alpha' \end{pmatrix}. \tag{5.44}$$

We now Fourier expand the second order perturbations and the sources:

$$\hat{\Sigma}^\mu(\tau,\sigma)=\sum_n \hat{\Sigma}^\mu_n(\tau)e^{-in\sigma}, \tag{5.45}$$

$$\hat{U}^\mu(\tau,\sigma)=\sum_n \hat{U}^\mu_n(\tau)e^{-in\sigma}. \tag{5.46}$$

Then, the matrix equation (5.39) reduces to:

$$\begin{pmatrix} \ddot{\hat{\Sigma}}{}^t_n \\ \ddot{\hat{\Sigma}}{}^r_n \\ \ddot{\hat{\Sigma}}{}^\varphi_n \end{pmatrix}+\mathcal{V}\begin{pmatrix} \hat{\Sigma}^t_n \\ \hat{\Sigma}^r_n \\ \hat{\Sigma}^\varphi_n \end{pmatrix}=\mathcal{G}^{-1}\begin{pmatrix} \hat{U}^t_n \\ \hat{U}^r_n \\ \hat{U}^\varphi_n \end{pmatrix}, \tag{5.47}$$

where:

$$\mathcal{V}=\mathcal{G}^{-1}(n^2I+\mathcal{F}-\mathcal{D}^2-\dot{\mathcal{D}}-2in\mathcal{E})\mathcal{G}$$
$$=\begin{pmatrix} n^2+m^2H^2\alpha'^2 & 0 & 2inmH\alpha' \\ 0 & n^2 & 0 \\ -2inmH\alpha' & 0 & n^2+m^2H^2\alpha'^2 \end{pmatrix}. \tag{5.48}$$

Thus, the second order perturbations are determined by a set of three coupled linear ordinary differential equations of second order with constant (matrix) coefficients and a complicated source-term. It follows that the complete solution is known explicitly: The matrix, Eq. (5.48), is diagonalized in the same way as in Eq. (5.29). The full solutions for the three second order perturbations are then written as free wave parts with frequencies $|n+mH\alpha'|$, $|n-mH\alpha'|$ and n, respectively, plus particular solutions involving integrals of the sources. This concludes the derivation of the second order perturbations.

Having calculated the first and second order perturbations, we can now also calculate the world-sheet energy-momentum tensor $T_{\pm\pm}$, Eqs. (5.12)–(5.15). This calculation is simplified using the fact that $T_{\pm\pm}$ are functions of $n(\sigma\pm\tau)$ while the first order perturbations η^μ are functions of $(n\sigma\pm|n\pm mH\alpha'|\tau)$. The first order perturbations can therefore only give constant contributions to $T_{\pm\pm}$. It is then straightforward to compute L_0 and $\tilde{L}_0$:

$$L_0=\pi\sum_n [(|n+mH\alpha'|+n)^2A_{1n}A^\dagger_{1n}$$
$$+(|n-mH\alpha'|+n)^2A_{2n}A^\dagger_{2n}]-\frac{\pi}{2}m^2\alpha'^2, \tag{5.49}$$

$$\tilde{L}_0=\pi\sum_n [(|n+mH\alpha'|-n)^2A_{1n}A^\dagger_{1n}$$
$$+(|n-mH\alpha'|-n)^2A_{2n}A^\dagger_{2n}]-\frac{\pi}{2}m^2\alpha'^2. \tag{5.50}$$

The constraints Eq. (5.14) for $n=0$ then become:

$$\sum_n n[|n+mH\alpha'|A_{1n}A^\dagger_{1n}+|n-mH\alpha'|A_{2n}A^\dagger_{2n}]=0, \tag{5.51}$$

as well as:

$$m^2\alpha'^2=2\sum_n [((n+mH\alpha')^2+n^2)A_{1n}A^\dagger_{1n}$$
$$+((n-mH\alpha')^2+n^2)A_{2n}A^\dagger_{2n}], \tag{5.52}$$

determining the mass of the string. Notice that the mass formula of the string is modified with respect to the usual flat spacetime expression ($m^2\alpha'^2=4\Sigma_n n^2[A_{1n}A_{1n}^\dagger+A_{2n}A_{2n}^\dagger]$). The reason for this modification is of course the presence of the cosmological constant through both gravity and torsion.

VI. THE QUANTUM MASS FORMULA

In this section we perform the canonical quantization using the results of the previous section. The first order comoving perturbations are described by the action [compare with Eqs. (5.22), (5.23)]:

$$S^{(2)}=-\frac{1}{4\pi\alpha'}\int d\tau d\sigma\,\eta^{ab}\delta_{RS}\times(\Phi^R_{,a}+A^R_{aU}\Phi^U)(\Phi^S_{,b}+A^S_{bV}\Phi^V), \tag{6.1}$$

where the vector-potential A^{RS}_a is anti-symmetric in the RS-indices, and explicitly given by:

$$A^{12}_\tau=A^{12}_\sigma=mH. \tag{6.2}$$

Again, it is interesting to compare with the analogue action in the absence of torsion [22]. In that case, the action for the comoving first order perturbations involved a *scalar* potential. Thus we see that the effect of the conformal invariance precisely is to cancel this scalar potential and replace it by a *vector* potential. This actually follows more generally from Eq. (5.9). The scalar potential comes from the $\overline{R}_{\mu\sigma\rho\lambda}$-term, while the vector potential comes from the term on the right hand side. Then, in the absence of torsion in AdS spacetime, the scalar potential survives but there is no vector potential. On the other hand, with torsion corresponding to conformal invariance in AdS spacetime, the vector potential survives, but there is no scalar potential since the torsion is parallelizing ($\overline{R}_{\mu\sigma\rho\lambda}=0$).

The momentum conjugate to Φ^R is:

$$\Pi_R\equiv\frac{\delta S^{(2)}}{\delta(\dot\Phi^R)}=\frac{1}{2\pi\alpha'}(\dot\Phi_R+mH\epsilon_{RS}\Phi^S);\quad \epsilon_{12}=1. \tag{6.3}$$

We now go directly to the quantum theory. The canonical commutation relations become:

$$[\Phi^R,\Phi^S]=[\Pi_R,\Pi_S]=0,$$
$$[\Pi_R,\Phi^S]=-i\delta^S_R\delta(\sigma-\sigma'). \tag{6.4}$$

It follows that the constants A_{Rn} and $\tilde{A}_{R_n}$ introduced in Eqs. (5.31)–(5.33), which are now considered as quantum operators, have the following commutation relations:

$$[A_{1n},A^\dagger_{1n}]=\frac{\alpha'}{2|n+mH\alpha'|},$$
$$[A_{2n},A^\dagger_{2n}]=\frac{\alpha'}{2|n-mH\alpha'|}. \tag{6.5}$$

It is convenient to make the redefinitions:

$$A_{1n}\equiv\begin{cases}\tilde{a}^1_n\sqrt{\dfrac{\alpha'}{2|n+mH\alpha'|}}, & n>0,\\ a^2_{-n}\sqrt{\dfrac{\alpha'}{2|n+mH\alpha'|}}, & n<0,\end{cases}$$
$$A_{2n}\equiv\begin{cases}\tilde{a}^2_n\sqrt{\dfrac{\alpha'}{2|n-mH\alpha'|}}, & n>0,\\ a^1_{-n}\sqrt{\dfrac{\alpha'}{2|n-mH\alpha'|}}, & n<0,\end{cases} \tag{6.6}$$

and similarly for the Hermitian conjugates. The a^R_n and $\tilde{a}^R_n$ represent conventionally normalized oscillators (no summation over R):

$$[a^R_n,(a^R_n)^\dagger]=[\tilde{a}^R_n,(\tilde{a}^R_n)^\dagger]=1\quad\text{for all}\quad n>0$$
$$[A^R_0,(A^R_0)^\dagger]=\frac{1}{2mH}. \tag{6.7}$$

The classical constraints $L_0=\tilde{L}_0=0$ in the quantum theory take the form:

$$(L_0-2\pi\alpha' a)|\psi\rangle=(\tilde{L}_0-2\pi\alpha' a)|\psi\rangle=0, \tag{6.8}$$

where a is the normal-ordering constant and the factor $2\pi\alpha'$ is introduced for later convenience. The normal-ordering constant is most easily obtained by symmetrization of the oscillator products in Eqs. (5.49), (5.50).

The physical state conditions Eq. (6.8), in terms of the conventionally normalized oscillators, then become:

$$m^2\alpha'=\sum_{n>0}\left[\frac{(n+mH\alpha')^2+n^2}{|n+mH\alpha'|}+\frac{(n-mH\alpha')^2+n^2}{|n-mH\alpha'|}\right]+\sum_{n>0}\left[\frac{(n+mH\alpha')^2+n^2}{|n+mH\alpha'|}\left((a^1_n)^\dagger a^1_n+(\tilde{a}^1_n)^\dagger\tilde{a}^1_n\right)\right]+\sum_{n>0}\left[\frac{(n-mH\alpha')^2+n^2}{|n-mH\alpha'|}\left((a^2_n)^\dagger a^2_n+(\tilde{a}^2_n)^\dagger\tilde{a}^2_n\right)\right], \tag{6.9}$$

and:

$$\sum_{n>0}n[(a^R_n)^\dagger a^R_n-(\tilde{a}^R_n)^\dagger\tilde{a}^R_n]=0, \tag{6.10}$$

where the zero-modes have been eliminated.

Equation (6.10) simply expresses that there must be an equal amount of left-movers and right-movers, so let us now consider the quantum mass formula Eq. (6.9) in a little more detail. The first term in Eq. (6.9) represents the zero-point energy. At the present stage it is formally infinite and need to be renormalized, but since it is just an overall constant, we skip it for the moment and concentrate on the oscillator parts of Eq. (6.9). As in Minkowski spacetime, it is convenient to characterize the physical states by the eigenvalue of the number-operator:

$$N=\frac{1}{2}\sum_{n>0}[(a_n^R)^\dagger a_n^R+(\tilde{a}_n^R)^\dagger \tilde{a}_n^R]. \tag{6.11}$$

Returning to Eq. (6.9), we first notice that we get the correct Minkowski spacetime result (as we should) in the limit $H=0$. Moreover, for the low-mass states ($mH\alpha' \ll 1$), the spectrum is just the Minkowski spectrum with small corrections of order $H^2\alpha'$ (we always assume $H^2\alpha' \ll 1$),

$$m^2\alpha'=4N+\mathcal{O}(H^2\alpha'); \quad \text{(low-mass states)}, \tag{6.12}$$

and we skipped the zero-point energy.

Consider now the high-mass states ($mH\alpha' \gg 1$). As an example, we consider the state:

$$[(\tilde{a}_1^1)^\dagger (a_1^1)^\dagger]^N|0\rangle, \tag{6.13}$$

for some large N (say) $N \gg (H^2\alpha')^{-1}$. This is a state with eigenvalue N of the number-operator, and its mass is approximately:

$$m^2\alpha' \approx 4H^2\alpha' N^2. \tag{6.14}$$

As another example of a high-mass state, consider (for N even):

$$[(\tilde{a}_2^1)^\dagger (a_2^1)^\dagger]^{N/2}|0\rangle. \tag{6.15}$$

This state also has eigenvalue N of the number-operator, but its mass is approximately:

$$m^2\alpha' \approx H^2\alpha' N^2. \tag{6.16}$$

More generally, we find for the high-mass states (up to a numerical factor):

$$m^2\alpha' \sim H^2\alpha' N^2 \quad \text{(high-mass states)}. \tag{6.17}$$

Notice that states with the same eigenvalue of the number-operator [for instance the states Eqs. (6.13), (6.15)], do not necessarily have the same mass. This is the case both for the low-mass states and the high-mass states. In the low-mass spectrum, the effect is just like a fine-structure, while in the high-mass spectrum, the states are completely mixed up. This is very different from the case of Minkowski spacetime where states with the same eigenvalue of the number-operator always have the same mass.

Finally, we should compare also with the results obtained for AdS spacetime but without torsion [22]. In that case, the mass formula was found to be:

$$m^2\alpha'=2\sum_{n>0}\frac{2n^2+m^2H^2\alpha'^2}{\sqrt{n^2+m^2H^2\alpha'^2}}+\sum_{n>0}\frac{2n^2+m^2H^2\alpha'^2}{\sqrt{n^2+m^2H^2\alpha'^2}} \times[(a_n^R)^\dagger a_n^R+(\tilde{a}_n^R)^\dagger \tilde{a}_n^R]. \tag{6.18}$$

It follows that the results agree in the low-mass spectrum and in the high-mass spectrum, while there is a minor difference in the intermediate region. In fact, as already discussed in connection with the frequencies of the first order perturbations [see the discussion after Eq. (5.36)], the effect of the conformal invariance is to ‘‘complete the squares.’’

It means that the conclusions obtained in Ref. [22] for AdS spacetime without torsion generally hold as well in the presence of torsion, corresponding to conformal invariance (parallelizing torsion). In particular, the scale of the high-mass states is set by the Hubble constant H (and not by α') as follows from Eq. (6.17).

Moreover, the level spacing corresponding to Eq. (6.17), grows proportionally to N:

$$\frac{d(m^2\alpha')}{dN} \propto N. \tag{6.19}$$

This is contrary to the case of Minkowski spacetime where it is constant. As shown in [22], this implies that the density of levels, $\rho(m)$ grows like $\sim e^{\sqrt{m/H}}$ and that the partition function for a gas of strings in AdS spacetime is well-defined for any temperature β^{-1} (with or without torsion). That is, there is no Hagedorn temperature in AdS spacetime.

It also follows from Eq. (6.17) that the entropy is proportional to $\sqrt{m}$:

$$S \sim \sqrt{N} \sim \sqrt{m}, \tag{6.20}$$

where we skipped numerical factors. It is interesting to notice that this result is formally similar to the recent results of the entropy obtained for the quantization of the $2+1$ BH-AdS spacetime [23].

Notice also that the results of the canonical quantization obtained in this section agree with the results obtained by semi-classical quantization of circular strings obtained in Sec. IV.

VII. CONCLUSIONS

In conclusion, we have considered classical and quantum strings in $2+1$ dimensional anti–de Sitter spacetime with parallelizing torsion, corresponding to the conformally invariant $SL(2,R)$-WZWN background.

By considering special and generic string configurations classically and quantum mechanically, we have extracted the *precise* effects of the conformal invariance, both on the classical dynamics of strings and on the quantum mass spectrum.

Generally, we have seen that the conformal invariance leads to a number of mathematical simplifications. On the other hand, the physical properties turn out to be more or less unchanged, as compared to the case without torsion.

This means that the original results obtained in AdS spacetime without torsion [13,16,22] still hold in the presence of conformal invariance, that is, in the presence of parallelizing torsion. At the quantum level, this means in particular that the high-mass states are governed by:

$$m \sim HN; \quad N \in N_0 \quad (N\ \text{‘‘large’’}), \tag{7.1}$$

where m is the string mass and H is the Hubble constant. It follows that the level spacing grows proportionally to N:

$$\frac{d(m^2\alpha')}{dN}\sim N, \tag{7.2}$$

while the entropy goes like:

$$S\sim\sqrt{m}. \tag{7.3}$$

Moreover, it follows that there is no Hagedorn temperature, so that the partition function is well-defined at any temperature.

These results were obtained using two independent methods, namely semi-classical quantization of circular oscillating strings and canonical quantization of string oscillator modes. The results of the two approaches agree and they agree with the results obtained for vanishing torsion [13,16,22].

The central charge in the $2+1$ AdS WZWN model takes the value [24,6,25]:

$$c=\frac{3k}{k-2}.$$

Conformal invariance thus holds for k such that $c=26$. This leads to the value for k:

$$k=\frac{52}{23}. \tag{7.4}$$

This means that conformal invariance holds provided the string tension and H are related as follows:

$$H=\frac{1}{\sqrt{k\alpha'}}=\sqrt{\frac{23}{52\alpha'}},$$

where we used Eqs. (2.10) and (7.4).

Moreover, the scalar curvature takes then the value

$$R=-\frac{6}{k\alpha'}=-\frac{69}{26\alpha'}.$$

It would be interesting to generalize our analysis to higher dimensional anti–de Sitter spacetime AdS_D. Contrary to the three dimensional case, AdS_D can generally not be described as a group-manifold. It can however be represented as a coset-space:

$$\mathrm{AdS}_D=\frac{SO(D-1,2)}{SO(D-1,1)}. \tag{7.5}$$

It follows that AdS_D is generally not conformally invariant. However, as shown in Refs. [24,6], conformal invariance can be achieved for certain values of the cosmological constant. More precisely, the central charge of the WZWN model of level k for the coset represented in Eq. (7.5) is [24,6,25]:

$$C=\frac{kD(D+1)}{2[k+1-D]}-\frac{kD(D-1)}{2[k+2-D]}. \tag{7.6}$$

As shown in Ref. [24], the condition of conformal invariance,

$$C=26, \tag{7.7}$$

has solutions in an *arbitrary* number of dimensions. However, in each dimension, the cosmological constant must take very specific values. That is, Eqs. (7.6), (7.7) lead to an equation of the form: $k=k(D)$ with solutions $D,k(D)$ for *arbitrary* values of D.

For our purposes, we would have to consider the gauged WZWN models corresponding to the coset-spaces Eq. (7.5) [19,26]. This would allow us to read off the corresponding metric and antisymmetric tensor, which are necessary for the investigation of string dynamics in these backgrounds. This is currently under investigation.

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[1] H. J. de Vega and N. Sánchez, Phys. Lett. B **197**, 320 (1987).

[2] H. J. de Vega and N. Sánchez, in *String Quantum Gravity and Physics at the Planck Scale*, Proceedings of the Erice Workshop held in June 1992, edited by N. Sanchez (World Scientific, Singapore, 1993), pp. 73–185, and references given therein; H. J. de Vega and N. Sánchez, in *String Gravity and Physics at the Planck Scale*, edited by N. Sánchez and A. Zichichi (NATO-Advanced Study Institute, Series C, Vol. 476) (Kluwer, Dordrecht, 1996), pp. 11–63; A. L. Larsen and N. Sánchez, *ibid.* pp. 65–103.

[3] M. Bañados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992).

[4] J. Balog, L. O'Raifeartaigh, P. Forgacs, and A. Wipf, Nucl. Phys. **B325**, 225 (1989).

[5] P. Petropoulos, Phys. Lett. B **236**, 151 (1990).

[6] I. Bars and D. Nemeschansky, Nucl. Phys. **B348**, 89 (1991).

[7] S. Hwang, Nucl. Phys. **B354**, 100 (1991).

[8] E. Witten, Commun. Math. Phys. **92**, 455 (1984).

[9] J. Horne and G. T. Horowitz, Nucl. Phys. **B368**, 444 (1992).

[10] W. Rindler, *Essential Relativity* (Springer-Verlag, New York, 1979), Chap. 8.11.

[11] N. Kaloper, Phys. Rev. D **48**, 2598 (1993).

[12] G. T. Horowitz and D. L. Welch, Phys. Rev. Lett. **71**, 328 (1993).

[13] A. L. Larsen and N. Sánchez, Phys. Rev. D **50**, 7493 (1994).
[14] R. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D **11**, 3424 (1975).
[15] H. J. de Vega and J. M. Maillet, Phys. Rev. D **28**, 1441 (1983).
[16] H. J. de Vega, A. L. Larsen, and N. Sánchez, Phys. Rev. D **51**, 6917 (1995).
[17] V. E. Zakharov and A. V. Mikhailov, Sov. Phys. JETP **47**, 1017 (1979).
[18] H. Eichenherr, in *Integrable Quantum Field Theories*, Tvärminne Proceedings, edited by J. Hietarinta and C. Montonen, Lecture Notes in Physics Vol. 151 (Springer-Verlag, Berlin, 1982).
[19] I. Bars and K. Sfetsos, Mod. Phys. Lett. A **7**, 1091 (1992).
[20] S. Kar, Phys. Rev. D **54**, 6408 (1996).
[21] C. Nash and S. Sen, *Topology and Geometry for Physicists* (Academic, London, 1987).
[22] A. L. Larsen and N. Sánchez, Phys. Rev. D **52**, 1051 (1995).
[23] A. Strominger, ''Black Hole Entropy from Near-Horizon Microstates,'' Report No. HUTP-97/A106, hep-th/9712251; D. Birmingham, I. Sachs, and S. Sen, ''Entropy of Three-Dimensional Black Holes in String Theory,'' hep-th/9801019; D. Birmingham, ''String Theory Formulation of anti-de Sitter Black Holes,'' hep-th/9801145.
[24] E. S. Fradkin and V. Y. Linetsky, Phys. Lett. B **261**, 26 (1991).
[25] P. Goddard, A. Kent, and D. Olive, Commun. Math. Phys. **103**, 105 (1986).
[26] I. Bars and K. Sfetsos, Phys. Lett. B **277**, 269 (1992).